MATH 3371 **Sample Exam 2**

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Name: KEY

Instructions:

- All answers must be written clearly.
- You may use a calculator, but you must show all your work in order to receive credit.
- Be sure to erase or cross out any work that you do not want graded.
- If you need extra space, you may use the back sides of the exam pages (if you do, please write me a note so that I know where to look).
- You must include all work to receive full credit.
- **It is not enough to only study this sample exam for the test. You should go over the Homework problems as well for extra practice**

1. A student claims that she can tell Friendly's ice cream from Herrell's ice cream. As a test, she is given ten samples of ice cream (each sample is either from Friendly's or Herrell's) and asked to identify each one. She is right eight times. What is the probability that she would be right exactly eight times if she guessed randomly for each sample?

$$X \sim Bin\left(10, \tfrac{1}{2}\right)$$

$$\mathbb{P}(X=8) = \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 = \frac{45}{2^{10}}$$

2. Let X represent the number of typos in Math books. Suppose that X is not Poisson but rather distributed with the following PMF

$$p_X(i) = \frac{2}{3^{i+1}}, i = 0, 1, 2, 3, \ldots$$

What is the expected number of errors in Math books?

$$\mathbb{E}X = \sum_{i=0}^{\infty} x_i\, p(x_i) = \sum_{i=0}^{\infty} i \cdot P_X(i)$$

$$= \sum_{i=0}^{\infty} i \cdot \frac{2}{3^{i+1}} = 0 \cdot \frac{2}{3} + 1 \cdot \frac{2}{3^2} + 2 \cdot \frac{2}{3^3} + 3 \cdot \frac{2}{3^4} + \cdots$$

$$= 1 \cdot \frac{2}{3^2} + 2 \cdot \frac{2}{3^3} + 3 \cdot \frac{2}{3^4} + \cdots$$

$$= \frac{2}{3^2}\left(1 + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3^2} + 4 \cdot \frac{1}{3^3} + \cdots\right) = \frac{2}{9}\left(1 + 2x + 3x^2 + \cdots\right)$$

$$= \frac{2}{9} \cdot \frac{1}{(1-\frac{1}{3})^2} = \frac{2}{9} \cdot \frac{1}{(\frac{2}{3})^2} = \boxed{\frac{1}{2}}$$

Recall,

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

differentiating

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

want to start w/ 1

w/ $\boxed{x = \frac{1}{3}}$

3. Births in a hospital occur randomly at an average rate of 1.8 births per hour.

(a) What is the probability of observing 4 births in a given hour at the hospital?

Let X = # of births in a given hour

Mean rate $\lambda = 1.8 \Rightarrow X \sim \text{Poisson}(1.8)$

$\therefore \mathbb{P}(X=4) = e^{-1.8}\frac{(1.8)^4}{4!} = \boxed{.07231}$

(b) What about the probability of observing more than or equal to 2 births in a given hour at the hospital?

$X \sim \text{Poisson}(1.8)$

$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X < 2) = 1 - (\mathbb{P}(X=0) + \mathbb{P}(X=1))$

$= 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) = 1 - e^{-1.8}\frac{(1.8)^0}{0!} - e^{-1.8}\frac{(1.8)^1}{1!}$

$= 1 - .16529 - .29753 = \boxed{.537}$

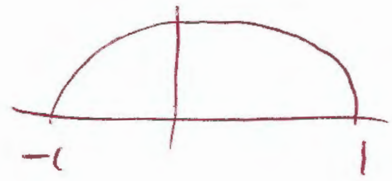
(c) What is the probability that we observe 5 births in a given 2 hour interval?

~~Let~~ X = # of births in a 2 hour interval

mean rate $\boxed{\lambda = 1.8 + 1.8 = 3.6}$ $X \sim \text{Poisson}(3.6)$

$\mathbb{P}(X=5) = e^{-3.6}\frac{(3.6)^5}{5!} = \boxed{.13768}$

$f(x) \to$



4. Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} c\left(1-x^6\right) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} c(1-x^6)\,dx = c\left[x - \frac{x^7}{7}\right]_{-1}^{1}$$

$$= c\left[\left(1-\frac{1}{7}\right) - \left(-1+\frac{1}{7}\right)\right] = c\cdot\frac{12}{7} \Rightarrow \boxed{c = \frac{7}{12}}$$

(b) What is the cumulative distribution function of X? Give the function, not just a sketch.

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx = \int_{-1}^{a} \frac{7}{12}(1-x^6)\,dx = \frac{7}{12}\left[x - \frac{x^7}{7}\right]_{-1}^{a}$$

$$= \frac{7}{12}\left[a - \frac{a^7}{7} + \frac{6}{7}\right] \quad \text{for} \quad -1 < a < 1$$

(c) What is $\mathbb{P}(-2 < X < 0)$?

Note since $X \in (-1, 1)$, then

$$\mathbb{P}(-2 < X < 0) = \mathbb{P}(X < 0) = F_X(0) = \frac{7}{12}\left[0 + \frac{6}{7}\right] = \boxed{\frac{1}{2}}$$

(d) What is $\mathbb{E}[X]$?

$$\mathbb{E}X = \int_{-1}^{1} x\cdot\frac{7}{12}(1-x^6)\,dx = \frac{7}{12}\left[\frac{x^2}{2} - \frac{x^8}{8}\right]_{-1}^{1} = \boxed{0}$$

(e) What is $Var(X)$?

$$\mathbb{E}X^2 = \int_{-1}^{1} x^2\,\frac{7}{12}(1-x^6)\,dx = \frac{7}{12}\left[\frac{x^3}{3} - \frac{x^9}{9}\right]_{-1}^{1} = \boxed{\frac{7}{27}}$$

$$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{7}{27} - 0^2 = \boxed{\frac{7}{27}}$$

5. Suppose that the time required to replace a car's windshield can be represented by an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$

(a) What is the probability that it will take at least 3 hours to replace a windshield?

Recall from class: If $X \sim exp(\lambda)$ then $\boxed{\mathbb{P}(X > a) = e^{-\lambda a}}$

Since $X \sim Exp(\frac{1}{2})$, then

$$\mathbb{P}(X > 3) = e^{-\frac{1}{2} \cdot 3} \approx .2231$$

(b) What is the probability that it will take at least 5 hours to replace a windshield given that it hasn't been finished after 2 hours?

Long way: Just compute $\mathbb{P}(X > 5 \mid X > 2)$

Short way: Recall from class that Exponential R.V.s are memoryless, thus

$$\mathbb{P}(X > 5 \mid X > 2) = \mathbb{P}(X > 2+3 \mid X > 2) = \mathbb{P}(X > 3) = e^{-\frac{3}{2}} \approx .2231$$

6. Suppose that the amount of time a repairman needs to fix a furnace is uniformly distributed between 1.5 and c hours. Suppose we know that 20 percent of the repairman's jobs take less than 3 hours to finish. Find the expected amount of time a repairman need to fix a furnace.

$X \sim U(1.5, c)$ so $f_X(x) = \begin{cases} \frac{1}{c-1.5}, & 1.5 < x < c \\ 0 & o/w \end{cases}$

We know that

$$.2 = \mathbb{P}(X < 3) = \int_{1.5}^{3} \frac{1}{c-1.5}\,dx = \frac{3-1.5}{c-1.5} = \frac{1.5}{c-1.5}$$

thus $c - 1.5 = \frac{1.5}{.2} = 7.5 \Rightarrow \boxed{c = 7.5 + 1.5 = 9}$

Thus $\mathbb{E}X = \frac{a+b}{2} = \frac{1.5+9}{2} = \boxed{5.25 \text{ hours}}$

7. Consider the random variable X with probability density function

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the p.d.f. of the random variable $Y = X^3$. What is the known distribution of X?

Step 1: Write the CDF of Y in terms of CDF of X

$$F_Y(x) = \mathbb{P}(Y \le x) = \mathbb{P}(X^3 \le x)$$
$$= \mathbb{P}(X \le \sqrt[3]{x}) = F_X(\sqrt[3]{x})$$

Step 2: Use fact that $\boxed{f_Y(x) = F_Y'(x)}$

$$f_Y(x) = F_Y'(x) = \frac{d}{dx}\left[F_X(x^{1/3})\right],$$ Use Chain Rule

$$= F_X'(x^{1/3}) \cdot (x^{1/3})' = f_X(x^{1/3}) \cdot \frac{1}{3x^{2/3}}$$

$$= \begin{cases} 3(x^{1/3})^2 \cdot \frac{1}{3x^{2/3}}, & 0 < x^{1/3} < 1 \\ 0, & o/w \end{cases} = \begin{cases} \cancel{3} \cdot \cancel{x^{2/3}} \cdot \frac{1}{\cancel{3}\cancel{x^{2/3}}}, & 0^3 < x^{3/3} < 1^3 \\ 0, & o/w \end{cases}$$

$$= \boxed{\begin{cases} 1, & 0 < x < 1 \\ 0, & ow \end{cases}}$$

$\therefore X \sim \text{Uniform}(0,1)$